- Euclidean Signature Spacetime

$$\underbrace{Z[J]}_{\text{Partition Function}} = e^{-W[J]} = \int \mathcal{D}\phi \, \exp\{-(S[\phi] - \phi\cdot J)\}$$

Generating Functional of connected correlators

Ex: $\cdot\ S[\phi] = \int d^dx \sqrt{g} \left\{ \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi + \frac{m^2}{2}\phi^2 + V(\phi) \right\}$

$$\phi\cdot J = \int d^dx \sqrt{g}\, \phi(x) J(x)$$

E.O.M: $\dfrac{\delta\{S[\phi] - \phi\cdot J\}}{\delta\phi} = 0$

$$\Big(-\underbrace{g_{\mu\nu}\nabla^\mu\nabla^\nu}_{\Box} + m^2\Big)\phi_b + V'(\phi_b) - J = 0$$

split the quantum field

$\phi = \phi_b + \eta \rightsquigarrow$ background field method

flutuations

$$\mathcal{D}\phi = \mathcal{D}\eta$$

$$\mathcal{L}[\phi_b + \eta] = \frac{1}{2}\nabla_\mu \phi_b \nabla^\mu \phi_b + \frac{m^2}{2}\phi_b^2 + V(\phi_b) - J\phi_b$$

$$\boxed{+ \nabla_\mu \phi_b \nabla^\mu \eta + m^2 \phi_b \eta + V'(\phi_b)\eta - J\eta}$$

$$\boxed{+ \frac{1}{2}\nabla_\mu \eta \nabla^\mu \eta + \frac{m^2}{2}\eta^2 + \frac{1}{2}V''(\phi_b)\eta^2} + \mathcal{O}(\eta^3)$$

linear: $\int d^dx \sqrt{g} \left\{ \nabla_\mu \phi_b \nabla^\mu \eta + m^2 \phi_b \eta + V'(\phi_b) \eta - J \eta \right\}$

$$= \int_M d^dx \sqrt{g} \left\{ \big(\underbrace{-\nabla_\mu \nabla^\mu \phi_b + m^2 \phi_b + V'(\phi_b) - J}_{=0 \,\rightsquigarrow\, \text{E.O.M}} \big) \eta \right\} + \text{b.t.}$$

quadractic: $\int d^dx \sqrt{g} \left\{ \frac{1}{2} \nabla_\mu \eta \nabla^\mu \eta + \frac{m^2}{2} \eta^2 + \frac{1}{2} V''(\phi_b) \eta^2 \right\}$

$$= \int d^dx \sqrt{g} \left\{ \frac{1}{2} \eta \big(\underbrace{-\Box + m^2 + V''(\phi_b)}_{D \,\rightsquigarrow\, \text{wave operator}} \big) \eta \right\} + \text{b.t.}$$

- $Z[J] = \int D\phi \exp\left\{ -(S[\phi] - J\cdot\phi) \right\}$

$$= \int D\eta \exp\left\{ -(S[\phi_b] - J\cdot\phi_b) + \int d^dx \sqrt{g} \left\{ -\frac{1}{2} \eta D \eta \right\} \right\}$$

$$= \exp\left\{ -(S[\phi_b] - J\cdot\phi_b) \right\} \cdot \left[\det(D) \right]^{-\frac{1}{2}}$$

* $\det(e^D) = e^{\mathrm{Tr}(D)}$

$$\prod_\lambda e^\lambda = e^{\sum_\lambda \lambda}$$

- $\left[\det(D)\right]^{-\frac{1}{2}} = \left[\det\left(e^{\ln(D)}\right)\right]^{-\frac{1}{2}} = \left[e^{\mathrm{Tr}(\ln(D))}\right]^{-\frac{1}{2}}$

$$= \exp\left\{ -\frac{1}{2} \mathrm{Tr}(\ln(D)) \right\}$$

$$\Rightarrow Z[J] = \exp\left\{ -\left(S[\phi_b] - J\cdot\phi_b + \frac{1}{2} \mathrm{Tr}(\ln(D)) \right) \right\}$$

$$W[J] = S[\phi_b] - J\cdot\phi_b + \frac{1}{2}\mathrm{Tr}(\ln(D))$$

$$\hookrightarrow \Gamma[\phi_b] = S[\phi_b] + \frac{1}{2}\mathrm{Tr}(\ln(D))$$

$$\hookrightarrow \Gamma_{1\text{-loop}} = \frac{1}{2}\mathrm{Tr}(\ln(D))$$

$$\text{“}= \frac{1}{2}\left(\sum_\lambda \ln(\lambda)\right)\text{”}$$

• Some heuristic arguments to the Heat Kernel Approach

$$\lambda > 0, \quad \ln(\lambda) = -\int_0^\infty \frac{dt}{t}\, e^{-t\lambda}$$

$$\frac{d}{d\lambda}(\dots) \Rightarrow \frac{1}{\lambda} = \cancel{-}\int_0^\infty \frac{dt}{\cancel{t}}\,(\cancel{-t})e^{-t\lambda} = \frac{e^{-t\lambda}}{-\lambda}\bigg|_0^\infty = -\frac{e^{-\infty}}{\lambda} + \frac{e^0}{\lambda}$$

$$\Rightarrow \ln(\lambda) = -\int_0^\infty \frac{dt}{t}\, e^{-t\lambda} + C$$

$\hookrightarrow \infty$, but λ-independent

$$\hookrightarrow \Gamma_{1\text{-loop}} = \text{“}-\frac{1}{2}\int_0^\infty \frac{dt}{t}\sum_\lambda e^{-t\lambda}\text{”}$$

$$= -\frac{1}{2}\int_0^\infty \frac{dt}{t}\,\mathrm{Tr}(e^{-tD}) = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\, K(t,D)$$

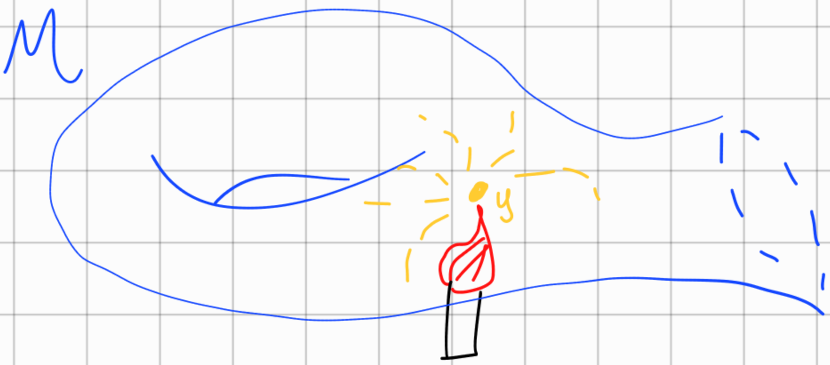
$$K(t,D) = \mathrm{Tr}\left(e^{-tD}\right) = \int d^d x \sqrt{g}\, \underbrace{K(t;x,x;D)}_{\text{Heat Kernel}}$$

↳ Heat Trace

- $(\partial_t + D_x)\, K(t;x,y;D) = 0\,, \quad K(0;x,y,D) = \delta(x,y)$

$$\begin{cases} (\partial_t + D_x)\, u(t,x) = 0 \\ u(0,x) = f(x) \end{cases}$$

$$\Rightarrow u(t,x) = \int_M d^d y\, K(t,x,y,D)\, f(y)$$



- $D^{-1}(x,y) = \int_0^\infty dt\, K(t;x,y;D)$

$$\begin{aligned} D_x D^{-1}(x,y) &= \int_0^\infty dt\, D_x K(t;x,y;D) \\ &= \int_0^\infty dt \left(-\frac{\partial}{\partial t} K(t;x,y;D)\right) \\ &= K(0;x,y;D) - \cancel{K(\infty;x,y;D)}^{\,0} \\ &= \delta(x,y) \end{aligned}$$

↳ Decay fast enough

Heat Kernel Expansion

- $K(t;D = -\nabla_\mu \nabla^\mu + m^2 + X)$

$$= \mathrm{Tr}\, e^{-t(-\Box + m^2 + X)}$$

$$= (4\pi t)^{-d/2} e^{-tm^2} \int d^d x \sqrt{g} \sum_{r=0}^{\infty} \mathrm{tr}\left(b_{2r}(x)\right) t^r$$

- $b_K(x)$ are known as the heat kernel coefficients

$$\Gamma_{1\text{-loop}} = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\, \mathrm{Tr}\, e^{-t(-\Box+m^2+X)}$$

$$= -\frac{1}{2}\frac{1}{(4\pi)^{d/2}}\int d^dx \sqrt{g} \sum_{r=0}^{\infty} \mathrm{tr}\, b_{2r}(x) \int_0^\infty dt\, t^{r-1-\frac{d}{2}} e^{-tm^2}$$

$$\int_0^\infty dt\, t^{r-1-\frac{d}{2}} e^{-tm^2} \xrightarrow{t'=tm^2,\ dt'=m^2dt} \frac{1}{\cancel{m^2}}\frac{1}{(m^2)^{\frac{2r-\cancel{2}-d}{2}}} \underbrace{\int_0^\infty dt'\, t'^{\overset{z}{\widetilde{r-\frac{d}{2}}}-1} e^{-t}}_{\Gamma(r-\frac{d}{2})} = \frac{1}{m^{2r-d}}\Gamma\left(r-\frac{d}{2}\right)$$

$$= -\frac{1}{2}\frac{1}{(4\pi)^{d/2}}\int d^dx \sqrt{g} \sum_{r=0}^{\infty} \frac{\Gamma(r-\frac{d}{2})}{m^{2r-d}}\, \mathrm{tr}\left(b_{2r}(x)\right)$$

Going to Lorentzian metric signature

$$x^0 \to i x^0 \quad , \quad W_{1\text{-loop}} \to -i\, W_{1\text{-loop}}$$

- $$\Gamma_{1\text{-loop}} = (-1)^F \frac{1}{2}\frac{1}{(4\pi)^{d/2}}\int d^dx \sqrt{g} \sum_{r=0}^{\infty} \frac{\Gamma(r-d/2)}{m^{2r-d}}\, \underbrace{\mathrm{tr}\left(b_{2r}(x)\right)}$$

For fermionic Fields

trace over internal indexes

- The Heat Kernel coeficients are made of monomials of gauge and geometric invariants

- R, $R_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$ and derivatives
- X and derivatives
- $\Omega_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$ and derivatives

$\hookrightarrow \nabla = \nabla^{[R]} + \omega$

$\hookrightarrow$ Covariant Curvature + Gauge derivatives

Ex: $\nabla_\mu - ieA_\mu$

- The heat Kernel coeficients are "universal" in the sence that they don't depend on the manifold / spacetime dimension or specif properties of D.

- The K-th heat Kernel coeficient is made of a specific combination of all possible independent invariants of mass dimension k, constructed from X, Ω, R and their derivatives.

$[X] = [\Omega] = [R] = 2$

$\hookrightarrow$ One cannot construct an odd-dimension invariant on a manifold without boundary

$\hookrightarrow$ $b_{2r+1} = 0$, $\forall r \geq 0$

$\rightsquigarrow$ For Manifolds with boundary we have another invariants related to extrinsic curvature of boundary and boundary conditions and we consider integrals over the boundary ∂M, which is a $d-1$ dimensional manifold. $\backsim$

- First proving this universal propertie of the heat kernel coefficients, one can choose simple Manifolds (like spheres, torus, products of manifolds, etc) or operators to compute each coefficient. Even so, it's a non trivial task to compute higher order coefficients. Only the first heat kernel coefficients are explicitly known.

↳ This was the Gilkey method (1975)
↳ There is also an iterative method (De Witt, 1965)

- At the end of the day, we have:

$$b_0 = I, \quad b_2 = \frac{1}{6} R I - X, \quad b_4 = \alpha_1 R^2 I + \alpha_2 R X + \alpha_3 R_{\mu\nu} R^{\mu\nu} I + \dots$$

Simple Example: $V(\phi) = \frac{\lambda}{4!} \phi^4 \quad (d=4)$

↳ $X = V''(\phi_{bg}) = \frac{\lambda}{2} \phi_{bg}^2, \quad \Omega_{\mu\nu} = 0, \quad R = R_{\mu\nu} = R_{\mu\nu\rho\sigma} = 0$

$$\Gamma_{1\text{-loop}} = \frac{1}{2} \frac{1}{(4\pi)^{d/2}} \int d^d x \sum_{r=0}^{\infty} \frac{\Gamma(r - d/2)}{m^{2r-d}} \operatorname{tr}(b_{2r}(x))$$

$$\Gamma = \Gamma_{\text{tree level}} + \Gamma_{1\text{-loop}} + \dots$$

$$= S[\phi_b] + \frac{1}{2} \frac{1}{(4\pi)^{d/2}} \int d^d x \sum_{r=0}^{\infty} \frac{\Gamma(r - d/2)}{m^{2r-d}} \operatorname{tr}(b_{2r}(x))$$

$$= \int d^d x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right\} + \frac{1}{2} \frac{1}{(4\pi)^{d/2}} \int d^d x \sum_{r=0}^{\infty} \frac{\Gamma(r - d/2)}{m^{2r-d}} \operatorname{tr}(b_{2r}(x))$$

$\Gamma(r-d/2) = \Gamma(r-2)$ is finite for $r \geq 3$

$$\begin{cases} b_2 \rightsquigarrow \text{Mass renormalization} \\ b_4 \rightsquigarrow \text{Coupling } (\lambda) \text{ renormalization} \end{cases}$$

$$\int d^dx \left\{ \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \delta m^2 \phi^2 \right\} + \frac{1}{2}\frac{1}{(4\pi)^{d/2}} \int d^dx \sum_{r=0}^{\infty} \frac{\Gamma(r-d/2)}{m^{2r-d}} \mathrm{tr}(b_{2r}(x))$$

$\boxed{b_2:}$
$$+ \int d^dx \left\{ -\frac{1}{2}m^2\phi^2 + \frac{1}{2}\frac{m^{d-2}}{(4\pi)^{d/2}}\Gamma(1-d/2)\left(-\frac{\lambda}{2}\phi^2\right) \right\} + \ldots$$

$[\lambda] = 4-d = \varepsilon$

$\lambda \to \mu^\varepsilon \lambda$ ← to keep λ admensional

- $$m^2 \frac{\mu^\varepsilon}{m^\varepsilon}\Gamma\left(1-\frac{d}{2}\right) \xrightarrow{\varepsilon \to 0} m^2 \frac{(-1)^{(1)}}{(1)!}\cdot 2\left(\frac{1}{\varepsilon} + \ln\left(\frac{\mu}{m}\right)\right)$$

$$= -m^2\left(\frac{2}{\varepsilon} + \ln\left(\frac{\mu^2}{m^2}\right)\right)$$

$$\left\{ -\frac{1}{2}m^2\phi^2 + \frac{1}{2}\frac{1}{(4\pi)^2}\left(-\frac{\lambda}{2}\phi^2\right)\left[-m^2\left(\frac{2}{\varepsilon} + \ln\left(\frac{\mu^2}{m^2}\right)\right)\right] \right\}$$

$$= \left\{ -\frac{1}{2}\Bigg[\underbrace{m^2 - \frac{\lambda m^2}{32\pi^2}\left(\frac{2}{\varepsilon} - \ln\left(\frac{m^2}{\mu^2}\right)\right)}_{m_R^2 = m^2 + \delta m^2}\Bigg]\phi^2 \right\}$$

$$\hookrightarrow \boxed{\delta m^2 = -\frac{\lambda m^2}{32\pi^2}\left(\frac{2}{\varepsilon} - \ln\left(\frac{m^2}{\mu^2}\right)\right)} + \text{higher order loops}$$

$$\int d^dx \left\{ \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right\} + \frac{1}{2}\frac{1}{(4\pi)^{d/2}} \int d^dx \sum_{r=0}^{\infty} \frac{\Gamma(r-d/2)}{m^{2r-d}} \operatorname{tr}(b_{2r}(x))$$

$\boxed{b_4:}$
$$+\int d^dx \left\{ -\frac{\lambda}{4!}\phi^4 + \frac{1}{2}\frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{m^{4-d}} \frac{\lambda^2}{8}\phi^4 \right\} + \dots$$

$b_4 = \frac{1}{2}X^2$

$$\frac{\mu^\epsilon}{m^\epsilon}\Gamma\left(2-\frac{d}{2}\right) \xrightarrow{\varepsilon \to 0} \frac{(-1)^0}{0!}\left(\frac{2}{\varepsilon} - \ln\left(\frac{m^2}{\mu^2}\right)\right)$$

$$\Rightarrow \left\{ -\frac{\lambda}{4!}\phi^4 + \frac{1}{2}\frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \ln\left(\frac{m^2}{\mu^2}\right)\right)\frac{\lambda^2}{8}\phi^4 \right\}$$

$$= -\frac{1}{4!}\left\{ \lambda - 4!\,\frac{1}{2}\cdot\frac{1}{(4\pi)^2}\frac{\lambda^2}{8}\left(\frac{2}{\epsilon} - \ln\left(\frac{m^2}{\mu^2}\right)\right) \right\}\phi^4$$

$$= -\frac{1}{4!}\left\{ \underbrace{\lambda - \frac{3}{16\pi^2}\lambda^2\cdot\left(\frac{1}{\varepsilon} + \ln\left(\frac{\mu}{m}\right)\right)} \right\}\phi^4$$

$$\lambda_R = \lambda + \delta\lambda, \quad \delta\lambda = -\frac{3}{16\pi^2}\lambda^2\cdot\left(\frac{1}{\varepsilon} + \ln\left(\frac{\mu}{m}\right)\right)$$

$$\Rightarrow \boxed{\beta(\lambda) = \mu\frac{\partial}{\partial\mu}(-\delta\lambda) = \frac{3}{16\pi^2}\lambda^2} + \text{higher order loops}$$

Advantages: • Manifestly covariant approach;
Does not make significant distintions between different spins, gauge groups, etc
• Useful in complicated geometries (with boundaries and singularities (Ads, black holes,..)
• No need to deal directly with Feynmann Diagrams, propagators in momentum space, etc

• Main Limitation
• Is not directly applicable beyond one-loop approximation (but it is doable!)

The heat Kernel method is also a natural tool to deal with Gravitational Effective Field Theories.

The integral in (1.18) may be divergent at both limits. Divergences at $t = \infty$ are caused by zero or negative eigenvalues of D. These are the infra red divergences. They will not be discussed in this section. We simply suppose that the mass m is sufficiently large to make the integral (1.18) convergent at the upper limit. Divergences at the lower limit cannot be removed in such a way. Let us introduce a cut off at $t = \Lambda^{-2}$.

$$W_\Lambda = -\frac{1}{2}\int_{\Lambda^{-2}}^{\infty} \frac{dt}{t} K(t, D)\,. \tag{1.20}$$

It is now easy to calculate the part of W_Λ which diverges in the limit $\Lambda \to \infty$:

$$\begin{aligned} W_\Lambda^{\rm div} = -(4\pi)^{-n/2} \int d^n x \sqrt{g} \Bigg[& \sum_{2(j+l)<n} \Lambda^{n-2j-2l} b_{2j}(x,x) \frac{(-m^2)^l l!}{n-2j-2l} \\ & + \sum_{2(j+l)=n} \ln(\Lambda)(-m^2)^l l!\, b_{2j}(x,x) + \mathcal{O}(\Lambda^0) \Bigg]\,. \end{aligned} \tag{1.21}$$

We see that the ultra violet divergences in the one-loop effective action are defined by the heat kernel coefficients $b_k(x,x)$ with $k \le n$.